Linear Algebra

UNIT-4

ORTHOGONALISATION, EIGEN VALUES & EIGEN VECTORS

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ORTHOGONAL BASES

BS Grewal : 2.13 to 2.15, 28.9
Gilbert: 3.4, 5.1, 5.2

A basis consisting of mutually orthogonal vectors

Orthonormal BASIS

A basis consisting of unit length, mutually orthogonal vectors

Q1. Is $\{(0,2), (2,0)\}$ orthonormal basis?

$$\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0 \longrightarrow \text{orthogonal}$$

length $\neq 1 \Rightarrow$ not orthonormal

ORTHOGONAL MATRIX

- A matrix with orthonormal columns is called Q $(m \geq n)$
- If $m = n$, the matrix is orthogonal

Properties of Q

1. If Q (square or rectangular) has orthonormal columns, then

$$Q^T Q = I$$

Let $Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$ $\quad Q^T = \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}$

$$Q^TQ = \begin{bmatrix} q_1^Tq_1 & q_1^Tq_2 & \cdots & q_1^Tq_n \\ \vdots & & & \\ q_n^Tq_1 & q_n^Tq_2 & \cdots & q_n^Tq_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

2. An orthogonal matrix is square matrix with orthonormal columns,

$$Q^T = Q^{-1}$$

3. If Q is a tall matrix,

$Q^T =$ left inverse of Q

$$Q^TQ = I$$

4. Multiplication by any Q preserves the length

$$\|x\| = \|Qx\|$$

5. Q preserves inner products and angles

$$(Qx)^T(Qy) = x^TQ^TQy = x^Ty$$

6. If $q_1, q_2 \dots q_n$ are orthonormal bases of R^n then any vector b in R^n can be expressed as

$$b = x_1 q_1 + x_2 q_2 + \dots + x_n q_n \longrightarrow (1)$$

To solve for x_1, multiply (1) by q_1^T

$$q_1^T b = x_1 q_1^T q_1$$

$$x_1 = \frac{q_1^T b}{q_1 q_1^T} = q_1^T b$$

$$x_1 = q_1^T b \leftarrow \text{projection of } b \text{ onto } q_1$$

$$b = (q_1^T b) q_1 + (q_2^T b) q_2 + \dots + (q_n^T b) q_n$$

Rectangular Matrices with Orthonormal Columns

If Q has orthonormal columns, least squares solution becomes easier

$$Qx = b \quad \text{where} \quad b \notin C(Q)$$

Recall least squares solution:

$$Q^T Q \hat{x} = Q^T b$$

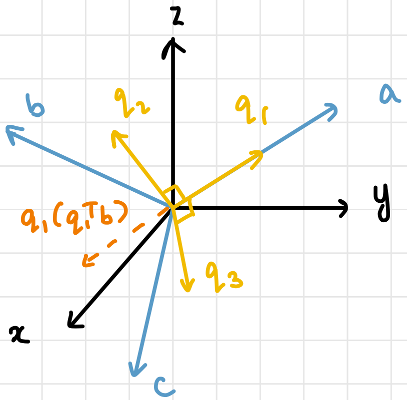
$$I \hat{x} = Q^T b$$

$$\boxed{\hat{x} = Q^T b}$$

GRAM-SCHMIDT PROCESS

Process of converting linearly independent vectors into a set of orthonormal vectors

Consider three linearly independent vectors a, b, c. The set of orthonormal vectors: q_1, q_2, q_3



$$q_1 = \frac{a}{\|a\|}$$

$$B = b - (q_1^T b) q_1$$

projection of b on q_1

$$q_2 = \frac{B}{\|B\|}$$

$$C = c - (q_2^T c) q_2 - (q_1^T c) q_1$$

$$q_3 = \frac{C}{\|C\|}$$

Projection of b onto a

$$p = a\,\frac{a^T b}{a^T a}$$

If b is not $\perp$ to a, the projection of b onto a must be subtracted to form an orthogonal vector to a

Q-R FACTORISATION

If $A_{m\times n}$ is a matrix with linearly independent columns, then A can be factorised as

$$A_{m\times n} = Q_{m\times n} R_{n\times n}$$

where Q is a matrix with orthonormal vectors (constructed using Gram-Schmidt Process) and R is an upper triangular and invertible matrix)

If $A = \begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix}$

We express a, b, c as linear combinations of q_1, q_2 and q_3

$$a = (q_1^T a) q_1$$

$$b = (q_1^T b) q_1 + (q_2^T b) q_2$$ ← $q_3 \perp$ ab plane

$$c = (q_1^T c) q_1 + (q_2^T c) q_2 + (q_3^T c) q_3$$

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}}_{Q_{m\times n}} \underbrace{\begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}}_{R_{n\times n}}$$

In general,

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 & \cdots & q_1^T a_n \\ 0 & q_2^T a_2 & \cdots & q_2^T a_n \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & q_m^T a_n \end{bmatrix}$$

System is Inconsistent – Least Squares Method

$Ax = b$ where $b \notin C(A)$

$$A^T A \hat{x} = A^T b$$

$$(QR)^T QR \hat{x} = (QR)^T b$$

$$R^T \underbrace{Q^T Q}_{I} R \hat{x} = R^T Q^T b$$

$$R^T R \hat{x} = R^T Q^T b$$

R is square matrix $\Rightarrow$ multiply by $(R^T)^{-1}$

$$\boxed{R\hat{x} = Q^T b}$$

Q2. Let $Q = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$

Verify that

(i) $Q^T Q = I$

(ii) $\|Qx\| = \|x\|$, $\|Qy\| = \|y\|$ or Q preserves length

(iii) $((Qx)^T Qy) = x^T y$

(i) $Q^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ $Q = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$

$$Q^T Q = \begin{bmatrix} 1/2 + 1/2 & 0 \\ 0 & 4/9 + 4/9 + 1/9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(ii) $Qx = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1-2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

$$\|Qx\| = \sqrt{9+1+1} = \sqrt{11}$$

$$\|x\| = \sqrt{2+9} = \sqrt{11}$$

$$Qy = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} = \begin{bmatrix} -3+4 \\ -3-4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}$$

$$\|Qy\| = \sqrt{1+49+4} = \sqrt{54}$$

$$\|y\| = \sqrt{18+36} = \sqrt{54}$$

(iii) $((Qx)^T Qy) = x^T y$

$$(Qx)^T Qy = x^T y$$

$$\begin{bmatrix} 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$$

$$[3+7+2] = [-6+18]$$

$$[12] = [12]$$

Q3. Find the orthogonal basis spanned by a set of vectors $a = (2,-5,1)$, $b = (4,-1,5)$

$$q_1 = \frac{a}{\|a\|} = \frac{(2,-5,1)}{\sqrt{4+25+1}} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$B = b - (q_1^T b) q_1 = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} - \frac{1}{\sqrt{30}} (8+5+5) \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} - \frac{1}{30}(18)\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} - 3/5 \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 14/5 \\ 2 \\ 22/5 \end{bmatrix} \qquad \|B\| = \sqrt{(14/5)^2 + 4 + (22/5)^2}$$

$$= \sqrt{\frac{156}{5}}$$

$$\|B\| = \frac{2\sqrt{195}}{5}$$

$$b = \frac{B}{\|B\|} = \frac{(14/5,\ 2,\ 22/5)}{2\sqrt{195}} \times 5$$

$$q_2 = \frac{1}{2\sqrt{195}}\begin{bmatrix} 14 \\ 10 \\ 22 \end{bmatrix}$$

Q4. Apply GS Process of Orthogonalisation to the vectors $a = (1,0,1)$, $b = (1,0,-1)$, $c = (0,3,4)$ to obtain an orthonormal basis q_1, q_2, q_3

$$q_1 = \frac{a}{\|a\|} = \frac{(1,0,1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$B = b - (q_1^T b)\, q_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 0$$

$$q_2 = \frac{(1, 0, -1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$C = c - (q_1^T c) q_1 - (q_2^T c) q_2$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - 2\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + 2\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$q_3 = \frac{C}{\|C\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Q5. $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$. Find q_1, q_2, q_3 orthonormal basis from a, b, c (columns of A).

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ a & b & c \end{array}$$

Then write A as QR

$a = (1, 0, 0)$

$$q_1 = \frac{a}{\|a\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$B = b - (q_1^T b)\, q_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \Rightarrow q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = c - (q_1^T c)\, q_1 - (q_2^T c)\, q_2$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A = QR$$

$$R = Q^T A$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Q6. Use GS Process to find a set of orthonormal vectors from the independent vectors

$a_1 = (1,0,1)$, $a_2 = (1,0,0)$, $a_3 = (2,1,0)$

Also find $A = QR$

Let orthonormal vectors be q_1, q_2, q_3

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{(1,0,1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$A_2 = a_2 - (q_1^T a_2) q_1$$

$$= (1,0,0) - 1/\sqrt{2}\,(1/\sqrt{2}, 0, 1/\sqrt{2})$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$q_2 = \frac{(1/2, 0, -1/2)}{\sqrt{1/4 + 1/4}} = \sqrt{2}\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$A_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - (2/\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - (2/\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$q_3 = \frac{A_3}{\|A_3\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \qquad Q^T = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = QR$$

$$Q^TA = R$$

$$R = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 2/\sqrt{2} \\ 0 & 1/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ 0 & 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Q7. Find a third column so that the matrix Q

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \_\_ \\ 1/\sqrt{3} & 2/\sqrt{14} & \_\_ \\ -1/\sqrt{3} & 3/\sqrt{14} & \_\_ \end{bmatrix}$$ is orthogonal

Assume (x, y, z) is third column

null space $\perp$ row space

$$A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow \quad \begin{aligned} R_1 &\to \sqrt{3}R_1 \\ R_2 &\to \sqrt{14}R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

one free variable $z \longrightarrow$ infinite solutions

can assume any value for z

let $z = 1$

$$x + y - 1 = 0 \Rightarrow x = 1 - y \longrightarrow (1)$$

$$x + 2y + 3 = 0 \Rightarrow x = -3 - 2y \rightarrow (2)$$

(1) & (2)

$$1 - y = -3 - 2y$$

$$2y - y = -3 - 1$$

$$y = -4 \Rightarrow x = 5$$

$$\therefore (x, y, z) = \frac{(5, -4, 1)}{\sqrt{25 + 16 + 1}}$$

$$= \frac{1}{\sqrt{42}} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & 5/\sqrt{42} \\ 1/\sqrt{3} & 2/\sqrt{14} & -4/\sqrt{42} \\ -1/\sqrt{3} & 3/\sqrt{14} & 1/\sqrt{42} \end{bmatrix}$$

Q5. Find an orthonormal set q_1, q_2, q_3 for which q_1 & q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$

(i) Which fundamental subspace contains q_3?

(ii) What is the least square solution of $Ax = b$ if $b = (0, 3, 0)$?

$$q_1 = \frac{(1, 2, 2)}{\sqrt{9}} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$B = b - (q_1^T b)\, q_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - (1/3 + 2 + 2/3) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$q_2 = \frac{(0, 1, -1)}{\sqrt{2}} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

ii) Left null space

$$A^T = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \qquad A^T y = 0$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Let $z = 1$ (free var)

$$x + 2y + 2 = 0$$

$$y - 1 = 0 \Rightarrow y = 1$$

$$x + 2 + 2 = 0 \Rightarrow x = -4$$

$$q_3 = \frac{(-4, 1, 1)}{\sqrt{18}} = \begin{bmatrix} -4/\sqrt{18} \\ 1/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

(ii) least square solution

$b = (0,3,0)$

$$A\hat{x} = b$$

$$R\hat{x} = Q^T b$$

$$Q^T b = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3/\sqrt{2} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3/\sqrt{2} \end{bmatrix}$$

$$\Big\downarrow R_2 \rightarrow \sqrt{2} R_2$$

$$\begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$2y = 3 \Rightarrow y = 3/2$$

$$3x + 9/2 = 2$$

$$3x = -5/2$$

$$x = -5/6$$

$$\hat{x} = \begin{bmatrix} -5/6 \\ 3/2 \end{bmatrix}$$

Q9. If W is the subspace spanned by the orthogonal vectors (2,5,-1), (-2,1,1), find the point in W closest to (1,2,3)

Let W = column space of A

$$A = \begin{bmatrix} 2 & -2 \\ 5 & 1 \\ -1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad A = QR$$

$$R = Q^T A$$

$$R\hat{x} = Q^T b \qquad p = A\hat{x}$$

$$q_1 = \frac{(2,5,-1)}{\sqrt{4+25+1}} = \begin{bmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix}$$

$$B = b - (q_1^T b) q_1$$

$$= \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} - 0 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{(-2,1,1)}{\sqrt{6}} = \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 5 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{30} & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

$$R\hat{x} = Q^T b$$

$$\begin{bmatrix} \sqrt{30} & 0 \\ 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{30} & 0 \\ 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9/\sqrt{30} \\ 3/\sqrt{6} \end{bmatrix}$$

$$\left[\begin{array}{cc:c} \sqrt{30} & 0 & 9/\sqrt{30} \\ 0 & \sqrt{6} & 3/\sqrt{6} \end{array}\right]$$

$R_1 \to \sqrt{30}\, R_1$
$R_2 \to \sqrt{6}\, R_2$

$$\left[\begin{array}{cc:c} 30 & 0 & 9 \\ 0 & 6 & 3 \end{array}\right]$$

$$30x = 9 \qquad 6y = 3$$

$$x = 3/10 \qquad y = 1/2$$

$$\hat{x} = \begin{bmatrix} 3/10 \\ 1/2 \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 2 & -2 \\ 5 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3/10 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 6/10 - 1 \\ 15/10 + 1/2 \\ -3/10 + 1/2 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$$

Q10. Find an orthonormal set q_1, q_2, q_3 for which q_1 & q_2 span the column space of A

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

(a) Which fundamental subspace contains q_3?

(b) What is the least square solution of $A\hat{x} = b$ if $b = (1, 2, 7)$

$$q_1 = \frac{(1,2,-2)}{\sqrt{4+4+1}} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \overset{q^T}{\begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix}} \overset{a_2}{\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}} \overset{q}{\begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \left(1/3 - 2/3 - 8/3 \right) \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$q_2 = \frac{(2,1,2)}{\sqrt{9}} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

(a) q_3 is in left null space; let $q_3 = (x, y, z)$

$$\begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} R_1 \to 3R_1 \\ R_2 \to 3R_2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 6 \end{bmatrix}$$

free

z is free: let $z=1$

$$-3y=-6$$
$$y=2$$

$$x+4-2=0$$
$$x=-2$$

$$q_3=\frac{(-2,2,1)}{3}=\begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$$

(b) $A=\begin{bmatrix}1&1\\2&-1\\-2&4\end{bmatrix}$ $\qquad b=\begin{bmatrix}1\\2\\7\end{bmatrix}$

$$A\hat{x}=b,\qquad R\hat{x}=Q^Tb$$

$$R=Q^TA=\begin{bmatrix}1/3&2/3&-2/3\\2/3&1/3&2/3\end{bmatrix}\begin{bmatrix}1&1\\2&-1\\-2&4\end{bmatrix}=\begin{bmatrix}3&-3\\0&3\end{bmatrix}$$

$$Q^Tb=\begin{bmatrix}1/3&2/3&-2/3\\2/3&1/3&2/3\end{bmatrix}\begin{bmatrix}1\\2\\7\end{bmatrix}=\begin{bmatrix}-3\\6\end{bmatrix}$$

$$R\hat{x}=Q^Tb$$

$$\begin{bmatrix}3&-3\\0&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-3\\6\end{bmatrix}$$

$3y=6 \Rightarrow y=2$ $\qquad$ $3x-6=-3 \Rightarrow 3x=3$

$$x=1$$

$$\hat{x}=\begin{bmatrix}1\\2\end{bmatrix}$$

Q11. What multiple of $a_1 = (2,2)$ should be subtracted from $a_2 = (4,0)$ for the result to be orthogonal to a_1? Factor $A = QR$ with orthonormal vectors in Q.

can do $(a_2 - ka_1) \cdot a_1 = 0$

$$a_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \overbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}^{\text{vector } \parallel \text{ to } a_1}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - 2\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - (1)\, a_1$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$\therefore$ multiple $= 1$

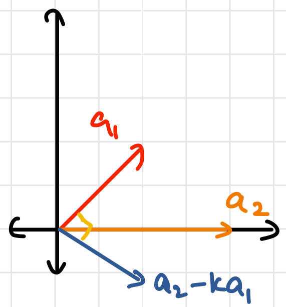
$$q_2 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}$$

OR

$$(a_2 - ka_1)^T a_1 = 0$$



$$\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} - k \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right)^T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4-2k \\ -2k \end{bmatrix}^T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0$$

$$2(4-2k) - 4k = 0$$

$$8 - 4k - 4k = 0$$

$$8k = 8$$

$$k = 1$$

EIGEN VALUES & EIGEN VECTORS

Let A be any square matrix of order n, then all the values of λ (real or complex) which satisfy the equation $|A-\lambda I| = 0$ are called the eigenvalues of A

$$|A-\lambda I| = 0 \longrightarrow \text{characteristic equation}$$

All the vectors 'x' that satisfy the equation $Ax = \lambda x$ or $(A-\lambda I)x = 0$ are called the eigen vectors corresponding to the eigen value λ

$$Ax = \lambda x \quad \text{or} \quad (A-\lambda I)x = 0$$

Note:

1. If A is a square matrix of order n, then there are exactly n eigenvalues of A

eg: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$ $\quad A-\lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$

$|A| = (1-\lambda)^2 - 1 = 0 \Rightarrow 1-\lambda = \pm 1$

$\lambda = 1 \pm 1$

$\lambda = 0, 2 \longrightarrow 2$ values

2. λ is an eigenvalue of A iff $A-\lambda I$ is singular, or $|A-\lambda I| = 0$

If $|A|$ is already 0, then $\lambda = 0$ is always an eigenvalue of A

3. If A is invertible, i.e $|A| \neq 0$, $\lambda = 0$ is never an eigenvalue of A

4. $(A - \lambda I)x = 0 \Rightarrow x \in N(A - \lambda I)$

5. If $Ax = \lambda x$ and $\lambda = 0$, then $Ax = 0$ and $x \in N(A)$

PROPERTIES OF EIGEN VALUES & EIGEN VECTORS

1. Given an eigen vector x of a matrix, corresponding eigen value λ is unique

2. Given an eigenvalue of a matrix, there are infinitely many eigenvectors

3. The eigenvalues of a square matrix and its transpose are equal

4. The eigenvalues of an idempotent matrix $(A^2 = A = A^n)$ are either 0 or 1

5. If λ is an eigenvalue of A with x as the corresponding eigen vector, then λ^2 is an eigen value of A^2 with the same eigen vector x

6. The trace of a matrix is equal to the sum of its eigenvalues (trace = sum of principal diagonal entries)

7. The product of all eigenvalues of A is the determinant of A

8. The eigenvalues of a triangular / diagonal matrix are the principal diagonal elements of the matrix

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

9. If λ is an eigenvalue of A and A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1}

10. If A is an orthogonal matrix, then if λ is an eigenvalue of A, $1/\lambda$ is also an eigenvalue of A

11. Cayley-Hamilton Theorem

Every square matrix A satisfies the characteristic equation

$$|A - \lambda I| = 0$$

Procedure

Step 1: Calculate $|A - \lambda I|$ (polynomial in λ of order n)

Step 2: Find roots of equation (eigenvalues)

Step 3: For each value of λ, solve the equation

$$(A - \lambda I)x = 0$$

Non-zero values of $x \rightarrow$ eigen vectors

Q12. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Verify that

(i) Trace = sum of eigenvalues

(ii) Determinant of A equals the product of eigenvalues

(iii) If we shift A to $A - 7I$

(a) what are the eigenvalues of $A - \lambda I$?

(b) How are they related to those of A?

characteristic eq.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) + 2 = 0$$

$$4 - 5\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \qquad \lambda^2 - (\text{trace})\lambda + \det = 0$$

$$\lambda = 2 \qquad \lambda = 3$$

Eigenvectors:

(a) $\lambda = 2$

$$(A-2I)x = 0$$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow \; R_2 \to R_2 + 2R_1$$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $y = k$

$$-x - k = 0$$
$$x = -k$$

$$x = \begin{bmatrix} -k \\ k \end{bmatrix} = \left\{ k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

(b) $\lambda = 3$

$$(A-3I)x = 0$$

$$\begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}$$

Let $y = k$

$$-2x - k = 0$$

$$x = -k/2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k/2 \\ k \end{bmatrix} = \left\{ k/2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$$

$$x = \left\{ c \begin{bmatrix} -1 \\ 2 \end{bmatrix} \mid c \in R \right\}$$

(i) Trace of A = sum of principal diagonals

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

eigen values

$$1 + 4 = 2 + 3 = 5$$

(ii) $|A| = \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 4 + 2 = 6$

product of λ's $= 2 \times 3 = 6$

(iii) $A \longrightarrow A - 7I$

$$A - 7I = \begin{bmatrix} 1-7 & -1 \\ 2 & 4-7 \end{bmatrix} = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix} = B$$

(a) Eigenvalues of B

$$|B-\lambda I| = 0$$

$$\begin{vmatrix} -6-\lambda & -1 \\ 2 & -3-\lambda \end{vmatrix} = 0$$

$$+(6+\lambda)(3+\lambda) + 2 = 0$$

$$\lambda^2 + 9\lambda + 18 + 2 = 0$$

$$\lambda^2 + 9\lambda + 20 = 0$$
$$(\lambda+4)(\lambda+5) = 0$$

$$\lambda = -4 \qquad \lambda = -5$$

(b) $\lambda_{A1} = 2 \qquad \lambda_{A2} = 3$

$\lambda_{B1} = -5 \qquad \lambda_{B2} = -4$

$$\lambda_{A1} - \lambda_{B1} = 2+5 = 7$$

$$\lambda_{A2} - \lambda_{B2} = 3+4 = 7$$

$$\therefore \lambda_A - \lambda_B = 7$$

$\therefore$ if $A \longrightarrow A + kI$ then $\lambda_k = k + \lambda$

Q13. Find the eigenvalues of the matrices A, A^2, A^{-1} and $A+4I$ given

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(i) $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$2-\lambda = \pm 1$$
$$\lambda = 2 \pm 1$$

$$\lambda_1 = 3, \quad \lambda_2 = 1$$

(ii) Property: $\lambda \to A \Rightarrow \lambda^2 \to A^2$

$$\lambda_1 = 9 \qquad \lambda_2 = 1$$

verify: $A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$

$$|A^2 - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)^2 = 16$$
$$5-\lambda = \pm 4 \Rightarrow \lambda = 1, 9$$

(iii) $1/\lambda$ is eigenvalue of A^{-1}

$\lambda_1 = 1 \quad \lambda_2 = 1/3$

(iv) $\lambda_1 = 5 \quad \lambda_2 = 7$

Q14. Write the 3 different 2x2 matrices for which eigenvalues are 4, 5 and $|A| = 20$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{aligned} ad - bc &= 20 \\ a + d &= 9 \end{aligned}$$

eg 1: $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$

eg 2: $\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$

eg 3: $\begin{bmatrix} 4 & 3 \\ 0 & 5 \end{bmatrix}$

Q15. Find the eigenvalues and eigenvectors for the given matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix}$$

$$= (2-\lambda)\big((3-\lambda)(2-\lambda)-2\big) - 2\big((2-\lambda)-1\big) + \big(2-(3-\lambda)\big)$$

$$= (2-\lambda)(\lambda^2-5\lambda+6-2) - 2(1-\lambda) + (\lambda-1)$$

$$= (2-\lambda)(\lambda^2-5\lambda+4) + 2\lambda - 2 + \lambda - 1$$

$$= 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda + 3\lambda - 3$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\lambda_1 = 5 \quad \lambda_2 = 1 \quad \lambda_3 = 1$$

(i) $\lambda = 5$

$$(A - 5I)x = 0$$

$$\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 1/3 R_1$

$R_3 \to R_3 + 1/3 R_1$

$$\begin{bmatrix} -3 & 2 & 1 \\ 0 & -4/3 & 4/3 \\ 0 & 8/3 & -8/3 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} -3 & 2 & 1 \\ 0 & -4/3 & 4/3 \\ 0 & 0 & 0 \end{bmatrix}$$

let $z = k$

$$-\frac{4}{3}y + \frac{4}{3}k = 0 \qquad\qquad -3x + 2k + k = 0$$

$$y = k \qquad\qquad x = k$$

$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = 1$

$$\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x + 2k_1 + k_2 = 0$$

$$x = -2k_1 - k_2$$

$$x = \left\{ \begin{bmatrix} -2k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}, k_1, k_2 \in R \right\}$$

$$x = \left\{ k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mid k_1, k_2 \in R \right\}$$

Q16. Find the eigenvalues and eigenvectors for the given matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$A = A^T \Rightarrow |A| = 0$ (product of λ's $= 0$)

$\therefore \lambda_1 = 0$

Eigenvalues

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)((7-\lambda)(3-\lambda) - 16) + 6(-6(3-\lambda)+8) + 2(24 + 2(\lambda - 7)) = 0$$

$$(8-\lambda)(\lambda^2 - 10\lambda + 21 - 16) + 6(-18 + 6\lambda + 8) + 2(24 + 2\lambda - 14) = 0$$

$$(8-\lambda)(\lambda^2 - 10\lambda + 5) + 6(6\lambda - 10) + 2(2\lambda + 10) = 0$$

$$\underline{8\lambda^2} - \underline{80\lambda} + 40 - \lambda^3 + \underline{10\lambda^2} - \underline{5\lambda} + \underline{36\lambda} - 60 + \underline{4\lambda} + 20 = 0$$

$$-\lambda^3 + 18\lambda^2 - 45\lambda + 0 = 0$$

$$\lambda(-\lambda^2 + 18\lambda - 45) = 0$$

$$-\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$-\lambda(\lambda - 15)(\lambda - 3) = 0$$

$\lambda = 0$
$\lambda = 3$
$\lambda = 15$

shortcut if $|A| = 0$

$$\frac{x}{\underbrace{y_1 z_2 - y_2 z_1}}, \quad \frac{y}{\underbrace{z_1 x_2 - z_2 x_1}}, \quad \frac{z}{\underbrace{x_1 y_2 - x_2 y_1}}$$

special sol.

$$\text{when } A = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(i) $\lambda = 0$

$$\frac{x}{24 - 14}, \quad \frac{y}{-12 + 32}, \quad \frac{z}{56 - 36}$$

$$\frac{x}{10}, \quad \frac{y}{20}, \quad \frac{z}{20}$$

$$x = \left\{ k \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\frac{x}{24-8}, \frac{y}{-12+20}, \frac{z}{20-36}$$

$$\frac{x}{16}, \frac{y}{8}, \frac{z}{-16}$$

$$x = \left\{ k \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \mid k \in R \right\}$$

(iii) $\lambda = 15$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

$$\frac{x}{24+16}, \frac{y}{12-28}, \frac{z}{56-36}$$

$$\frac{x}{40}, \frac{y}{-40}, \frac{z}{20}$$

$$x = \left\{ k \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

Q17. Find the eigenvalues and eigenvectors for the given matrices

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad |A| = 2+0+0-0-1-1 = 0$$

$$|A - \lambda I| = 0$$

$$\begin{array}{cc|ccc|cc} & & - & & & + & \\ & & 1-\lambda & -1 & 0 & & \\ & -1 & -1 & 2-\lambda & -1 & -1 & \\ -1 & 1-\lambda & 0 & -1 & 1-\lambda & 0 & -1 \end{array} = 0$$

$$(1-\lambda)(2-\lambda)(1-\lambda) + (0) + (0) - (1-\lambda)(-1)(-1) - (-1)(-1)(1-\lambda) + 0 = 0$$

$$(1-\lambda)(\lambda^2 - 3\lambda + 2) - (1-\lambda) - (1-\lambda) = 0$$

$$\lambda^2 - 3\lambda + 2 - \lambda^3 + 3\lambda^2 - 2\lambda - 1 + \lambda - 1 + \lambda = 0$$

$$-\lambda^3 + 4\lambda^2 - 3\lambda + 0 = 0$$

$$-\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$-\lambda(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$
$$\lambda_3 = 0$$

Eigenvectors

(i) $\lambda = 3$

$$(A - 3I)x = 0$$

$$\begin{bmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & -1 \\ 0 & -1 & 1-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - 1/2\, R_1$$

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$-\frac{1}{2}y - k = 0$ $\qquad\qquad$ $-2x + 2k = 0$

$y = -2k$ $\qquad\qquad$ $x = k$

$$x = \left\{ k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \middle|\, k \in R \right\}$$

(ii) $\lambda = 1$

$$\begin{bmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 1-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$-y = 0$

$y = 0$

$-x - k = 0$

$x = -k$

$$x = \left\{ k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} , k \in R \right\}$$

(iii) $\lambda = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$y - k = 0$
$y = k$

$x - k = 0$
$x = k$

$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

Q18. Find the eigenvalues and eigenvectors for the given matrices

(i) $A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(ii) $A_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

(i) $A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ $\quad |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 6\lambda + 5 - 1) - 1(1-\lambda-3) + 3(1 - 3(5-\lambda)) = 0$$

$$\lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + 2 + \lambda + 3 - 9(5-\lambda) = 0$$

$$-\lambda^3 + 7\lambda^2 + 0\lambda - 36 = 0$$

$\lambda_1 = -2$
$\lambda_2 = 6$
$\lambda_3 = 3$

Eigenvectors

(a) $\lambda = -2$

$$\begin{bmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 1/3 R_1} \begin{bmatrix} 3 & 1 & 3 \\ 0 & 20/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$y = 0$

$3x + 3k = 0$

$x = -k$

$$x = \left\{ k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \middle| k \in R \right\}$$

(b) $\lambda = 6$

$$\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3/5 R_1]{R_2 \to R_2 + 1/5 R_1} \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4/5 & 8/5 \\ 0 & 8/5 & -16/5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4/5 & 8/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$-\frac{4}{5} y + \frac{8}{5} k = 0$

$y = 2k$

$-5x + 2k + 3k = 0$

$x = k$

$$x = \left\{ k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

(c) $\lambda = 3$

$(A - 3I)x = 0$

$$\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3/2 R_1]{R_2 \to R_2 + 1/2 R_1} \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5/2 & 5/2 \\ 0 & 5/2 & 5/2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5/2 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$y = -k$

$-2x - k + 3k = 0$

$-2x + 2k = 0$

$x = k$

$$x = \left\{ k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, k \in R \right\}$$

(ii) $A_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

(ii) $A_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

Eigenvalues $|A_2 - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ -1 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-\lambda(-1-\lambda)) + 1(-1-\lambda) + 1(1-\lambda) = 0$$

$$(1-\lambda)(\lambda+\lambda^2) \cancel{-1} - \lambda \cancel{+1} - \lambda = 0$$

$$\lambda + \lambda^2 - \lambda^2 - \lambda^3 - 2\lambda = 0$$

$$-\lambda^3 - \lambda = 0$$

$$-\lambda(\lambda^2+1) = 0$$

$$\lambda = 0 \qquad \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

Eigenvectors

i) $\lambda = 0$

$$Ax = 0$$

$$\begin{array}{ccc} & & & & \text{free} \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

let $z = k$

$$y - k = 0$$
$$y = k$$

$$x - k + k = 0$$
$$x = 0$$

$$x = \left\{ k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

(ii) $\lambda = i$

$$(A - iI)x = 0$$

$$\begin{bmatrix} 1-i & -1 & 1 \\ 1 & -i & 0 \\ -1 & 1 & -1-i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

RAYLEIGH'S POWER METHOD

To find numerically largest / dominant eigenvalue and the corresponding eigenvector of a given matrix

Procedure

1. Start with the initial approximation

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then

$$A x_0 = \lambda_1 x_1$$

$$A x_1 = \lambda_2 x_2$$

$$A x_2 = \lambda_3 x_3$$

$$\vdots$$

Repeat until $x_n - x_{n-1}$ becomes negligible

$(\lambda_n \approx \lambda_{n-1})$

Q19. Calculate 5 iterations of the power method to find the dominant eigenvalue of A.

Use $x_0 = (1, 0, 0)$ as initial approximation

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$$

(i) $Ax_0 = \lambda_1 x_1$

Let $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Ax_0 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = \lambda_1 x_1$$

numerically largest value $= 4 = \lambda_1$

$$Ax_0 = 4 \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix} = \lambda_1 x_1 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

(ii) $Ax_1 = \lambda_2 x_2$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \end{bmatrix} \Rightarrow \lambda_2 = 5$$

$$Ax_1 = 5 \begin{bmatrix} 1 \\ 4/5 \\ -4/5 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1 \\ 4/5 \\ -4/5 \end{bmatrix}$$

(iii) $Ax_2 = \lambda_3 x_3$ $\quad\quad 4 + 4/5 + 4/5$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4/5 \\ -4/5 \end{bmatrix} = \begin{bmatrix} 28/5 \\ 28/5 \\ -26/5 \end{bmatrix} = \frac{28}{5} \begin{bmatrix} 1 \\ 1 \\ -13/14 \end{bmatrix}$$

$$\lambda_3 = \frac{28}{5} = 5.6 \qquad x_3 = \begin{bmatrix} 1 \\ 1 \\ -13/14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -0.93 \end{bmatrix}$$

(iv) $Ax_3 = \lambda_4 x_4$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -13/14 \end{bmatrix} = \begin{bmatrix} 83/14 \\ 83/14 \\ -79/14 \end{bmatrix} = \frac{83}{14} \begin{bmatrix} 1 \\ 1 \\ -79/83 \end{bmatrix}$$

$$\lambda_4 = \frac{83}{14} = 5.93 \qquad x_4 = \begin{bmatrix} 1 \\ 1 \\ -79/83 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -0.95 \end{bmatrix}$$

(v) $Ax_4 = \lambda_5 x_5$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -79/83 \end{bmatrix} = \begin{bmatrix} 494/83 \\ 494/83 \\ -478/83 \end{bmatrix} = \frac{494}{83} \begin{bmatrix} 1 \\ 1 \\ -478/494 \end{bmatrix}$$

$$\lambda_5 = \frac{494}{83} = 5.95 \qquad x_5 = \begin{bmatrix} 1 \\ 1 \\ -478/494 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -0.97 \end{bmatrix}$$

The largest eigenvalue is 6 and corresponding eigenvector is $x = (1, 1, -1)$

Q20. Obtain the numerically smallest eigenvalue of $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ starting with $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We know that if λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}

$\therefore$ smallest eigenvalue of A = reciprocal of largest eigenvalue of A^{-1}

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \qquad A^{-1} = \frac{1}{6-5}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

(i) $A^{-1}x_0 = \lambda_1 x_1$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -5/3 \end{bmatrix}$$

$$\lambda_1 = 3 \qquad x_1 = \begin{bmatrix} 1 \\ -5/3 \end{bmatrix}$$

(ii) $A^{-1}x_1 = \lambda_2 x_2$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -5/3 \end{bmatrix} = \begin{bmatrix} 14/3 \\ -25/3 \end{bmatrix} = \frac{14}{3}\begin{bmatrix} 1 \\ -25/14 \end{bmatrix}$$

$$\lambda_2 = \frac{14}{3} = 4.67 \qquad x_2 = \begin{bmatrix} 1 \\ -25/14 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.79 \end{bmatrix}$$

(iii) $A^{-1}x_2 = \lambda_3 x_3$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -25/14 \end{bmatrix} = \begin{bmatrix} 67/14 \\ -60/7 \end{bmatrix} = \frac{67}{14}\begin{bmatrix} 1 \\ -120/67 \end{bmatrix}$$

$$\lambda_3 = \frac{67}{14} = 4.79 \qquad x_3 = \begin{bmatrix} 1 \\ -120/67 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.79 \end{bmatrix}$$

(iv) $A^{-1}x_3 = \lambda_4 x_4$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -120/67 \end{bmatrix} = \begin{bmatrix} 321/67 \\ -575/67 \end{bmatrix} = \frac{321}{67}\begin{bmatrix} 1 \\ -575/321 \end{bmatrix}$$

$$\lambda_4 = 4.79 \qquad x_4 = \begin{bmatrix} 1 \\ -1.79 \end{bmatrix}$$

$\therefore$ smallest eigenvalue of $A = 1/\lambda_4 = 0.208$

verify:

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \qquad |A - 0.21I| \approx 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 5 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 - 5 = 0$$

$$\lambda^2 - 5\lambda + 1 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25-4}}{2} \qquad \frac{5-\sqrt{21}}{2} = 0.209$$

DIAGONALISATION of A MATRIX

- Suppose $A_{n \times n}$ has n linearly independent eigenvectors (not a defective matrix — order ≠ independent vectors)
- If these eigenvectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ (lambda)
- The eigenvalues of A are on the diagonal of Λ
- Λ is called the eigenvalue matrix and S is called the eigenvector matrix
- S is not unique
- Any matrix with distinct eigenvalues can be diagonalised

Proof

Let $x_1, x_2, \ldots, x_n$ be the independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

Let $S = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$

$$AS = A \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$$

$$AS = \begin{bmatrix} | & | & & | \\ Ax_1 & Ax_2 & \cdots & Ax_n \\ | & | & & | \end{bmatrix}$$

We know $Ax_1 = \lambda x_1$ as $(A-\lambda)x = 0$

$$AS = \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \lambda_n \end{bmatrix}$$

$$\boxed{AS = S\Lambda} \qquad \boxed{S^{-1}AS = \Lambda}$$

Note:

- $A = S\Lambda S^{-1}$
- $A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2 S^{-1}$ — all $\lambda \to \lambda^2$, same x

  $A^n = S\Lambda^n S^{-1} \quad \forall\ n \in \mathbb{Z}^+$
- $A^k = S\Lambda^k S^{-1} \longrightarrow 0$ as $k \to \infty$ if $|\lambda_i| < 1$

Q21. Show that $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalisable

Eigenvalues $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 4\lambda + 3) - 3(0) + 0 = 0$$

$$(1-\lambda)((1-\lambda)(3-\lambda) = 0$$

$\lambda = 1 \qquad \lambda = 3$

$\therefore$ only 2 independent eigenvectors

(defective matrix)

Q22. Check if $A = \begin{bmatrix} -8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalisable. If yes, find S.

eigenvalues

$$\begin{vmatrix} -8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(-8-\lambda)\left((\lambda^2-10\lambda+21)-16\right) + 6\left(6(\lambda-3)+8\right) + 2\left(24+2(\lambda-7)\right) = 0$$

$$-(\lambda+8)(\lambda^2-10\lambda+5) + 6(6\lambda-10) + 2(2\lambda+10) = 0$$

$$-\lambda^3 + 10\lambda^2 - 5\lambda - 8\lambda^2 + 80\lambda - 40 + 36\lambda - 60 + 4\lambda + 20 = 0$$

$$-\lambda^3 + 2\lambda^2 + 115\lambda - 80 = 0$$

$$\left.\begin{aligned} \lambda_1 &= -10.13 \\ \lambda_2 &= 11.44 \\ \lambda_3 &= 0.69 \end{aligned}\right\} \text{approx}$$

Eigenvectors

(i) $\lambda = -10.13$

$$x = \left\{ k \begin{bmatrix} -20.36 \\ -6.90 \\ 1 \end{bmatrix} \right\}$$

(ii) $\lambda = 11.44$

$$x = \left\{ k \begin{bmatrix} 0.65 \\ -1.78 \\ 1 \end{bmatrix} \right\}$$

(iii) $\lambda = 0.69$

$$x = \left\{ k \begin{bmatrix} -0.13 \\ 0.51 \\ 1 \end{bmatrix} \right\}$$

$$S = \begin{bmatrix} -20.36 & 0.65 & -0.13 \\ -6.90 & -1.78 & 0.51 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -10.13 & 0 & 0 \\ 0 & 11.44 & 0 \\ 0 & 0 & 0.69 \end{bmatrix}$$

Eigenvectors from https://www.emathhelp.net/calculators/linear-algebra/eigenvalue-and-eigenvector-calculator/?i=%5B%5B-8%2C-6%2C2%5D%2C%5B-6%2C7%2C-4%5D%2C%5B2%2C-4%2C3%5D%5D

Cayley-Hamilton Theorem

Every square matrix A satisfies the characteristic equation

$$|A-\lambda I|=0$$

Replace λ with A in polynomial, solve for A^{-1}

Q23. Find the matrix A whose eigenvalues are 2 & 5 and eigenvectors are (1,0) and (1,1) using $S \Lambda S^{-1}$

$$A = S \Lambda S^{-1}$$

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

Q24. Factor $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ into $S \Lambda S^{-1}$ and hence compute A^{85}

Eigenvalues

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4\lambda + 4 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 3 \qquad \lambda = 1$$

Eigenvectors

(i) $\lambda = 3$

$$(A - 3I)x = 0$$

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$y = k \qquad -x + k = 0$

$\qquad\qquad x = k$

$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = 1$

$$(A-I)x = 0$$

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = k \qquad x + k = 0$$
$$x = -k$$

$$x = \left\{ k \begin{bmatrix} -1 \\ 1 \end{bmatrix}, k \in R \right\}$$

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$S \Lambda S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A^{85} = S \Lambda^{85} S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{85} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3^{85} & -1 \\ 3^{85} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3^{85}+1 & 3^{85}-1 \\ 3^{85}-1 & 3^{85}+1 \end{bmatrix}$$

Q25. Find $S \Lambda S^{-1}$ for given matrix $A = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -6 \\ 2 & -6-\lambda \end{vmatrix} = 0$$

$$(\lambda+6)(\lambda-1) + 12 = 0$$

$$\lambda^2 + 5\lambda - 6 + 12 = 0$$
$$\lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda+2)(\lambda+3) = 0$$

$$\lambda = -2 \qquad \lambda = -3$$

(i) $\lambda = -2$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1+2 & -6 \\ 2 & -6+2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2/3 R_1} \begin{bmatrix} 3 & -6 \\ 0 & 0 \end{bmatrix}$$

$$y = k$$

$$3x - 6k = 0$$
$$x = 2k$$

$$x = \left\{ k \begin{bmatrix} 2 \\ 1 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = -3$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1+3 & -6 \\ 2 & -6+3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 1/2 R_1} \begin{bmatrix} 4 & -6 \\ 0 & 0 \end{bmatrix}$$

$$y = k \qquad 4x - 6k = 0$$

$$x = \frac{3}{2} k$$

$$x = \left\{ c \begin{bmatrix} 3 \\ 2 \end{bmatrix}, c \in R \right\}$$

$$S = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A = S \Lambda S^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -9 \\ -2 & -6 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -8+9 & 12-18 \\ -4+6 & 6-12 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = A$$

Q26. Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ compute A^6

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-4)+2=0$$

$$\lambda^2-5\lambda+4+2=0$$

$$\lambda^2-5\lambda+6=0$$

$$(\lambda-3)(\lambda-2)=0$$

$$\lambda=3 \qquad \lambda=2$$

(i) $\lambda=3$

$$(A-3I)x=0$$

$$\begin{bmatrix} 1-3 & 1 \\ -2 & 4-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-2x+y=0$$
$$x=\frac{y}{2}$$

$$x = \left\{ k\begin{bmatrix} 1 \\ 2 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = 2$

$$\begin{bmatrix} 1-2 & 1 \\ -2 & 4-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 1/2 R_1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-x + y = 0$$
$$x = y$$
$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix}, k \in R \right\}$$

$$S = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad S^{-1} = -1\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\Lambda^2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\Lambda^6 = \begin{bmatrix} 3^6 & 0 \\ 0 & 2^6 \end{bmatrix}$$

$$S\Lambda^6 S^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3^6 & 0 \\ 0 & 2^6 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^6 & 2^6 \\ 2\times3^6 & 2^6 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2^7 - 3^6 & 3^6 - 2^6 \\ 2^7 - 2\times3^6 & 2\times3^6 - 2^6 \end{bmatrix}$$

$$= \begin{bmatrix} -601 & 665 \\ -1330 & 1394 \end{bmatrix}$$

Q27. Find all eigenvalues & eigenvectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and write 2 diff diagonalising matrices S.

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2-2\lambda+1-1) - 1(1-\lambda-1) + 1(1-1+\lambda) = 0$$

$$(1-\lambda)(\lambda^2-2\lambda) + \lambda + \lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + \lambda^2 - 2\lambda + 2\lambda = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$-\lambda^2(\lambda-3) = 0$$

$$\lambda = 0 \qquad \lambda = 3$$

only 2 eigenvalues

$\therefore$ cannot be diagonalised

Q28. Find the characteristic equation and hence find the inverse of A using Cayley-Hamilton Theory

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$|A - \lambda I| = 0$

$\lambda^3 - \text{trace}(A)\,\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0$

$\text{trace}(A) = 1 + 2 + 1 = 4$

$M_{11} = 2 - 6 = -4$

$M_{22} = 1 - 1 = 0$

$M_{33} = 2 - 12 = -10$

$\det(A) = 1(2-6) - 3(4-3) + 1(8-2)$
$= -4 - 3 + 6$
$= -1$

eq: $\lambda^3 - 4\lambda^2 - 14\lambda + 1 = 0$

$= A^3 - 4A^2 - 14A + I = 0$

multiply by A^{-1} on the right

$$A^2 - 4A - 14I + A^{-1} = 0$$

$$A^{-1} = -A^2 + 4A + 14I$$

$$A^2 = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 11 & 11 \\ 15 & 22 & 13 \\ 10 & 9 & 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -14 & -11 & -11 \\ -15 & -22 & -13 \\ -10 & -9 & -8 \end{bmatrix} + \begin{bmatrix} 4 & 12 & 4 \\ 16 & 8 & 12 \\ 4 & 8 & 4 \end{bmatrix} + \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 4 & 1 & -7 \\ 1 & 0 & -1 \\ -6 & -1 & 10 \end{bmatrix}$$

Q29. $A = \begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$

eq: $\lambda^3 - \text{trace}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A)$

$$A^2 = \begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$$

$$\text{trace}(A) = 1 + 2 + 3 = 6$$

$$= \begin{bmatrix} 20 & 3 & 8 \\ 27 & 13 & 18 \\ 20 & 5 & 19 \end{bmatrix}$$

$$M_{11} = 6 - 0 = 6$$
$$M_{22} = 3 - 10 = -7$$
$$M_{33} = 2 - 9 = -7$$

$$M_{11} + M_{22} + M_{33} = -8$$

$$\det(A) = 1(6-0) - 1(27-0) + 2(0-10)$$

$$= 6 - 27 - 20 = -41$$

eq: $A^3 - 6A^2 - 8A + 41I = 0$

multiply by A^{-1} to the right

$$A^2 - 6A - 8I + 41A^{-1} = 0$$

$$A^{-1} = \frac{-1}{41}\left(A^2 - 6A - 8I\right)$$

$$= \frac{-1}{41}\left(\begin{bmatrix} 20 & 3 & 8 \\ 27 & 13 & 18 \\ 20 & 5 & 19 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix} - 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{-1}{41}\left(\begin{bmatrix} 6 & -3 & -4 \\ -27 & -7 & 18 \\ -10 & 5 & -7 \end{bmatrix}\right) = \frac{1}{41}\begin{bmatrix} -6 & 3 & 4 \\ 27 & 7 & -18 \\ 10 & -5 & 7 \end{bmatrix}$$

Q30. Use CH Theorem to calculate A^{-1}

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$$

eq: $\lambda^3 - \text{trace}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A)$

$$0 = \lambda^3 - 7\lambda^2 + (9 + 6 + 0)\lambda - \left(1(9) - 2(3) + 2(1+2)\right)$$

$$0 = \lambda^3 - 7\lambda^2 + 15\lambda - 9$$

$$0 = A^3 - 7A^2 + 15A - 9I$$

multiply by A^{-1} to the right

$$A^2 - 7A + 15I - 9A^{-1} = 0$$

$$A^{-1} = \frac{1}{9}\left(A^2 - 7A + 15I\right)$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 8 \\ 4 & 5 & -4 \\ -4 & 4 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\left(\begin{bmatrix} 1 & 8 & 8 \\ 4 & 5 & -4 \\ -4 & 4 & 13 \end{bmatrix} - 7\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix} + 15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{9}\left(\begin{bmatrix} 9 & -6 & -6 \\ -3 & 6 & 3 \\ 3 & -3 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2/3 & -2/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & -1/3 & 0 \end{bmatrix}$$